International Journal of Theoretical Physics, Vol. 36, No. 10, 1997

Two-Dimensional R^n-Gravitation

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Received February 24, 1997

A system with constraints is considered: a string theory whose Lagrangian is the nth power of the Gauss curvature of a space-time manifold ($n \in N$, $n > 1$). The problem is solved exactly because after the constraints are utilized we deal with a variational problem with a trivial Lagrangian, i.e., its Euler–Lagrange equations are satisfied identically. One can say that the constraints "swallow" all dynamical degrees of freedom of the field theory. The investigation is a continuation of the 1989 work of Burlankov and Pavlov, who solved the problem of two-dimensional R^2-gravitation under the gauge $\gamma = 1$.

1. A SYSTEM WITH CONSTRAINTS

Recently, theoretical physicists have focused considerable interest on one-dimensional theories, e.g., the theory of strings (Green *et al.*, 1987). Studying model theories helps in understanding the peculiarities of systems with constraints such as Einstein's theory of gravitation. Of particular interest is the problem of introducing gravitation in one-dimensional space (e.g., Polyakov, 1987).

Insofar as the Hilbert functional of gravitation in $(1 + 1)$-dimensional space-time gives the Gauss–Bonnet topological invariant, we take as a Lagrangian the Gauss curvature in nth power, $n \in N$, with $n > 1$. The theory keeps its covariance. Although the calculations are cumbersome, the problem can be solved fully. It turns out to be a useful example to demonstrate the characteristics of systems with constraints.

We analyze the dynamics of a space-time metric taken in Arnowitt–Deser–Misner (ADM) form:

$$(g_{\mu\nu}) = \begin{pmatrix} \alpha^2 + \beta^2 & \gamma\beta \\ \gamma\beta & \gamma^2 \end{pmatrix}, \qquad \sqrt{g} = \alpha\gamma \tag{1.1}$$

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where the metric functions $\alpha(t, x)$ and $\beta(t, x)$ have meaning as Lagrange multipliers (to be shown below). The very simple form of the metric is in conformal gauge that is popular in string theory. But then one has to add lost constraints since a restriction to some coordinate system leads the corresponding Lagrange multiplier metric coefficients to vanish before we vary them.

The Gauss curvature in ADM coordinates can be expressed by (Pogorelov, 1969)

$$R = -\frac{1}{2\alpha^3\gamma^2}\det\begin{pmatrix} \alpha & \beta & \gamma \\ \dot{\alpha} & \dot{\beta} & \dot{\gamma} \\ \alpha' & \beta' & \gamma' \end{pmatrix}$$

$$- \frac{1}{2\alpha\gamma}\left(\left[\frac{(\gamma^2)^{\cdot} - (\gamma\beta)'}{\alpha\gamma}\right]^{\cdot} - \left[\frac{(\gamma\beta)^{\cdot} - (\alpha^2 + \beta^2)'}{\alpha\gamma}\right]'\right) \tag{1.2}$$

The functional of action is in the form

$$S = \frac{1}{2}\int_{t,x} R^n\alpha\gamma$$

$$= \frac{1}{2}\int_{t,x} (\alpha\gamma)^{1-n}\left[\left(\frac{\beta' - \dot{\gamma}}{\alpha}\right)^{\cdot} + \left(\frac{\beta\dot{\gamma}}{\alpha\gamma}\right)' - \left(\frac{(\alpha^2 + \beta^2)'}{2\alpha\gamma}\right)'\right]^n \tag{1.3}$$

where $\int_{t,x}$ denotes integration over the space-time manifold. Varying S by the metric, one gets the Euler–Lagrange equations:

$$\frac{\partial L}{\partial \alpha} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\alpha}} - \frac{\partial}{\partial x}\frac{\partial L}{\partial \alpha'} + \frac{\partial^2}{\partial x^2}\frac{\partial L}{\partial \alpha''} = 0$$

$$\frac{\partial L}{\partial \beta} - \frac{\partial}{\partial x}\frac{\partial L}{\alpha\beta'} + \frac{\partial^2}{\partial t \partial x}\frac{\partial L}{\partial \dot{\beta}'} + \frac{\partial^2}{\partial x^2}\frac{\partial L}{\partial \beta''} = 0$$

$$\frac{\partial L}{\partial \gamma} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\gamma}} - \frac{\partial}{\partial x}\frac{\partial L}{\partial \gamma'} + \frac{\partial^2}{\partial t^2}\frac{\partial L}{\partial \ddot{\gamma}} + \frac{\partial^2}{\partial t \partial x}\frac{\partial L}{\partial \dot{\gamma}'} = 0 \tag{1.4}$$

where L is the density of the Lagrange function.

The differential equations of the extremals (1.4) are very complicated. Matters are clearer in the Hamiltonian formulation, since we deal with a nondegenerate theory with higher derivatives (Dubrovin *et al.*, 1986), so it is assumed that the Hamiltonian description can be used. For this purpose we use a slightly modified version of the Ostrogradski method (Dubrovin *et* *al.*, 1986). It is relevant to introduce, along with generalized coordinates (α, β, γ),

$$u \equiv \frac{\beta' - \dot{\gamma}}{\alpha} \tag{1.5}$$

Then the action in coordinates (α, β, γ, u) takes the form

$$S = \frac{1}{2}\int_{t,x} (\alpha\gamma)^{1-n}\left[\dot{u} - \left(\frac{\beta u + \alpha'}{\gamma}\right)'\right]^n \tag{1.6}$$

Momentum densities are calculated using

$$\pi_u \equiv \frac{\delta S}{\delta \dot{u}} = \frac{\partial L}{\partial \dot{u}} = \frac{n}{2}(\alpha\gamma)^{1-n}\left[\dot{u} - \left(\frac{\beta u + \alpha'}{\gamma}\right)'\right]^{n-1}$$

$$\pi_\alpha \equiv \frac{\delta \mathrm{S}}{\delta \dot{\alpha}} = \frac{\partial \mathrm{L}}{\partial \dot{\alpha}}$$

$$\pi_\beta \equiv \frac{\delta \mathrm{S}}{\delta \dot{\beta}} = -\frac{\partial}{\partial \mathrm{x}}\frac{\partial \mathrm{L}}{\partial \dot{\beta}'}$$

$$\pi_\gamma \equiv \frac{\delta \mathrm{S}}{\delta \dot{\gamma}} = \frac{\partial \mathrm{L}}{\partial \dot{\gamma}} - \frac{\partial}{\partial \mathrm{t}}\frac{\partial \mathrm{L}}{\partial \ddot{\gamma}} - \frac{\partial}{\partial \mathrm{x}}\frac{\partial \mathrm{L}}{\partial \dot{\gamma}'} \tag{1.7}$$

The Hamiltonian

$$H = \int_x (\pi_u \dot{u} + \pi_\alpha \dot{\alpha} + \pi_\beta \dot{\beta} + \pi_\gamma \dot{\gamma} - L[\alpha, \dot{\alpha}, \alpha', \alpha''; \beta, \beta', \dot{\beta}', \beta''; \gamma, \dot{\gamma}, \gamma', \ddot{\gamma}, \dot{\gamma}']) \tag{1.8}$$

becomes, taking account of (1.5),

$$H = \int_x (\pi_u \dot{u} + \pi_\alpha \dot{\alpha} + \pi_\beta \dot{\beta} + \pi_\gamma(\beta' - \alpha u) - L[\alpha, \alpha', \alpha''; \beta, \beta'; \gamma, \gamma'; u, \dot{u}, u']) \tag{1.9}$$

where $\dot{u}$, by virtue of the nondegenerate Lagrangian, is expressed through π_u from (1.7). As a result, ignoring boundary terms, one has the following expression for the Hamiltonian:

$$H = \int_x \left(\pi_\alpha \dot{\alpha} + \pi_\beta \dot{\beta} + \alpha\left[(n-1)\left(\frac{2}{n^n}\right)^{1/(n-1)} \gamma \pi_u^{n/(n-1)} - u\pi_\gamma + \left(\frac{\pi_u'}{\gamma}\right)'\right] + \beta\left[-u\left(\frac{\pi_u'}{\gamma}\right) - \pi_\gamma'\right]\right) \tag{1.10}$$

Along with the equations of motion obtained by varying the Hamiltonian by variables $u(t, x)$, $\gamma(t, x)$, there are two differential constraints $\delta S/\delta\alpha$ and $\delta S/\delta\beta$:

$$(n-1)\left(\frac{2}{n^n}\right)^{1/(n-1)} \gamma\pi_u^{n/(n-1)} - u\pi_\gamma + \left(\frac{\pi_u'}{\gamma}\right)' = 0$$
$$u\left(\frac{\pi_u'}{\gamma}\right) + \pi_\gamma' = 0 \tag{1.11}$$

In relativistic theories their energy (the value of the Hamiltonian) vanishes.

The system (1.11) can be integrated once, and then takes the form

$$\frac{1}{2n-1}\left(\frac{2}{n}\right)^{n/(n-1)} \pi_u^{(2n-1)/(n-1)} + \left(\frac{\pi_u'}{\gamma}\right)^2 + \pi_\gamma^2 = \mathrm{c}(t)$$
$$u\pi_u' + \gamma\pi_\gamma' = 0 \tag{1.12}$$

where $c(t)$ is an arbitrary function of time.

The Hamiltonian formulation is defined by the Poisson structure $\hat{J}$ on the functional phase space $U(u, \pi_u, \gamma, \pi_\gamma)$. Its nonzero brackets are

$$\{\gamma(t, x), \pi_\gamma(t', x')\} = \delta(t - t')\delta(x - x')$$
$$\{u(t, x), \pi_u(t', x')\} = \delta(t - t')\delta(x - x') \tag{1.13}$$

(In geometrodynamics, U could be considered the phase space of Wheeler–De Witt superspace.)

Then we construct on the basis of constraints (1.11) the functionals

$$\Phi[\phi] = \int_{t,x} \left(\frac{1}{2n-1}\left(\frac{2}{n}\right)^{n/(n-1)} \pi_u^{(2n-1)/(n-1)} + \left(\frac{\pi_u'}{\gamma}\right)^2 + \pi_\gamma^2 - c(t)\right)\phi(t, x)$$
$$\Xi[\chi] = \int_{t,x} (u\pi_u' + \gamma\pi_\gamma')\chi(t, x) \tag{1.14}$$

and calculate their Poisson bracket

$$\{\Phi, \Xi\} = \int_{t,x;t',x'} \frac{\delta\Phi}{\delta z}\hat{J}\frac{\delta\Phi}{\delta z} \tag{1.15}$$

The result of the calculation is

$$\{\Phi[\phi], \Xi[\chi]\} = \Phi[(\phi\chi)'] + \int c(t)\phi(t, x) \tag{1.16}$$

So, the differential constraints (1.11) form a closed algebra (so there are no other constraints in the theory) and they do not annul the Poisson brackets. We can express the variables π_γ and u from the constraints as

$$\pi_\gamma^2 = c(t) - \frac{1}{2n-1}\left(\frac{2}{n}\right)^{n/(n-1)} \pi_u^{(2n-1)/(n-1)} - \left(\frac{\pi_u'}{\gamma}\right)^2$$

$$u = -\gamma\left(\frac{\pi_\gamma'}{\pi_u'}\right) \tag{1.17}$$

2. FUNCTIONAL FORMS

For the investigation of covariant theories, mathematical tools of the theory of variational complexes (Olver, 1988) that is a generalization of the De Rham complex of differential forms prove to be useful. The variational complexes are decomposed into two components. The first part is obtained by reformulation of the De Rham complex onto spaces of differential functions set on $V \subset X \times D$, where X is a space of independent variables and U is a space of dependent variables. A differential r-form is given by

$$\omega^r = \sum_J P_J[u]\, dx^J \tag{2.1}$$

where the P_J are differential functions and

$$dx^J = dx^{j_1} \wedge \ldots \wedge dx^{j_r}, \qquad 1 \leq j_1 < \ldots < j_r \leq p \tag{2.2}$$

constitute the basis of a space of differential r-forms $\wedge_r T^*X$.

Since for relativistic theories a consequence of covariance of the description is that the Hamiltonian is zero, we will be interested here only in the second part of the variational complex (Burlankov and Pavlov, 1989). Let us suppose the Hamiltonian constraints to be resolved.

Differential forms are active on "horizontal" variables X from M, and vertical forms are constructed analogously—they active on "vertical" variables u and their derivatives. A vertical k-form is a finite sum

$$\hat{\omega}^k = \sum P_J^\alpha[u]\, du_{J_1}^{\alpha_1} \wedge \ldots \wedge du_{J_k}^{\alpha_k} \tag{2.3}$$

where the P_J^α are differential functions. Here independent variables are like parameters.

Insofar as the vertical form $\hat{\omega}$ is built on a space of finite jets $M^{(n)}$, a vertical differential has properties of bilinearity, antidifferentiation, and closure, like an ordinary differential.

Here we use functional forms, connected with the introduced vertical forms, as functionals connected with differential functions. Let $\omega^k = \int_x \hat{\omega}^k$

be a functional k-form corresponding to a vertical k-form $\hat{\omega}^k$. A variational differential of a form ω^k is a functional $(k + 1)$-form corresponding to a vertical differential of a form $\hat{\omega}^k$:

$$\omega^{(k+1)} \equiv \delta\omega^k = \int_x \hat{d}\hat{\omega}^k \tag{2.4}$$

The basic properties are deduced from the properties of the vertical differential, so we get a variational complex. A variational differential defines an exact complex

$$0 \to \Lambda^0_* \xrightarrow{\delta} \Lambda^1_* \xrightarrow{\delta} \Lambda^2_* \xrightarrow{\delta} \Lambda^3_* \xrightarrow{\delta} \ldots \tag{2.5}$$

on spaces of functional forms on M.

Of particular interest in theoretical physics problems are functional forms: ω^0, ω^1, ω^2. In the present problem, after the constraints (1.14) are utilized, we get a functional 1-form, as a generalization of a differential Cartan form for dynamical systems:

$$\omega^1 = \int_{t,x} \left[\pi_\gamma\left(t, \pi_u, \left(\frac{\pi_u'}{\gamma}\right)\right) d\gamma - u\left(t, \pi_u, \left(\frac{\pi_u'}{\gamma}\right), \left(\frac{\pi_u'}{\gamma}\right)'\right) d\pi_u\right] \tag{2.6}$$

Equations of motion are obtained as a condition of the closedness of the 1-form: $\delta\omega^1 = 0$. But, as we demonstrate below, there is a 0-form ω^0:

$$\omega^0 = \int_{t,x} \hat{\omega}^0(t, \gamma, \pi_u) \tag{2.7}$$

so that $\delta\omega^0 = \omega^1$, i.e., ω^1 is not only a closed form, it is an exact one.

Acting by the operator of the variational differential δ on the form (2.7), we get

$$\delta\omega^0 = \int_{t,x} \left[\frac{\delta\omega^0}{\delta\gamma} d\gamma + \frac{\delta\omega^0}{\delta\pi_u} d\pi_u\right] \tag{2.8}$$

From this we find conditions on $\hat{\omega}^0$ (t, γ, π_u) are

$$\frac{\partial\hat{\omega}^0}{\partial\gamma} = \pi_\gamma\left(t, \pi_u, \left(\frac{\pi_u'}{\gamma}\right)\right)$$

$$\frac{\partial\hat{\omega}^0}{\partial\pi_u} - \frac{\partial}{\partial_x}\left(\frac{\partial\hat{\omega}^0}{\partial\pi_u'}\right) = -u\left(t, \pi_u, \left(\frac{\pi_u'}{\gamma}\right), \left(\frac{\pi_u'}{\gamma}\right)'\right) \tag{2.9}$$

The system (2.9) can be solved analytically:

$$\hat{\omega}^0 = \int_{t,x} \gamma \left[c(t) - \frac{1}{2n-1} \left(\frac{2}{n}\right)^{n/(n-1)} \pi_u^{(2n-1)/(n-1)} - \left(\frac{\pi_u'}{\gamma}\right)^2 \right]^{1/2}$$
$$+ \int_{t,x} \pi_u' \arcsin \left[\frac{\pi_u'}{\gamma} \left(c(t) - \frac{1}{2n-1} \left(\frac{2}{n}\right)^{n/(n-1)} \pi_u^{(2n-1)/(n-1)} \right)^{-1/2} \right] \tag{2.10}$$

where $\pi_u(\alpha, \dot{\alpha}, \alpha', \alpha''; \beta, \beta', \dot{\beta}', \beta''; \gamma, \dot{\gamma}, \ddot{\gamma}, \dot{\gamma}')$ in the initial variables is

$$\pi_u = \frac{1}{\alpha\gamma} \left[\left(\frac{\beta' - \dot{\gamma}}{\alpha} \right)^{\cdot} + \left(\frac{2\beta\dot{\gamma} - (\alpha^2 + \beta^2)'}{2\alpha\gamma} \right)' \right] \tag{2.11}$$

The formula (2.10) generalizes a result obtained in Burlankov and Pavlov (1989).

We get a generalized De Rham variational complex:

$$0 \to \Lambda_*^0 \xrightarrow{\delta} \Lambda_*^1 \xrightarrow{\delta} 0 \tag{2.12}$$

because the operator of the variational differential δ is nilpotent: $\delta^2 = 0$. So the generalized De Rham cohomology group is trivial. Translating into physical language, we conclude that the functional of action (1.3) does not define any dynamical problem.

ACKNOWLEDGMENTS

The research described in this publication was made possible in part by grants from the Ministry of General and Special Education of the Russian Federation, and Udmurt State University.

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